Methods of Proof for Boolean Logic

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Announcements

For 02.12

1. HW 4 due next Monday (02.17)

Outline

The Big Picture

Where is Today?

- You are taking a logic class
- Logic is mainly about *logical consequence*
  - It's about conclusions following (or not following) from premises
- So far, we've explored two methods for understanding logical consequence:
  1. Proof (Ch.2: informal, formal)
  2. Tautological Consequence (Ch.4: truth tables)
- However, we have not discussed how the methods of proof can be used for the Booleans
- That will be our project for today

The Big Picture
But Wait...

- We had to learn truth tables, why proofs too?
- Truth tables provide powerful logical tools for the Booleans, but have two significant limitations:
  1. **Impractical**: Truth tables get extremely large as they get more interesting. An interesting scientific argument could have well over 14 atomic sentences. That would take a table with over 16,000 rows!
  2. **Limited Applications**: As we learned yesterday, there are logical consequences that aren't tautological consequences. That's because truth tables are blind to the logic of expressions other than the Booleans, such as =.
- The methods of proof fill this gap admirably

Proofs
The Way Forward

- We've studied the basics of proofs but we haven't done any proofs involving the Boolean connectives
- Today we'll be discussing the informal methods of proof for the Booleans

¡+-¿ Next week we'll extend our formal proof system ($\mathcal{F}$) with formal rules for the Boolean connectives
  - These formal rules will closely mirror the informal proof methods discussed today

Proof
What is it?

Proof

A proof is a step-by-step demonstration which shows that a conclusion C must be true in any circumstance where some premises, say P_1 and P_2 are true

1. The step-by-step demonstration of C can proceed through intermediate conclusions
2. It may not be obvious how to show C from P_1 and P_2, but it may be obvious how to show C from some other claim Q that **is** an obvious consequence of P_1 and P_2
3. Each step must provide incontrovertible evidence for the next

Proof
Steps, What?

The Nature of Steps

Each step of a proof involves an appeal to certain facts about the meaning of the vocabulary involved. These facts are what we implicitly appeal to when we say a step is *obvious*

- What kind of facts?
- Facts which guarantee that the step will never lead us from something true to something false
- Let's consider an old example

Proof
An Old Inference Step

Indiscernibility of Identicals

If n is m, then whatever is true of n is also true of m (where 'n' and 'm' are names)

- This is a fact about the meaning of *is*
- This fact guarantees that if it is true that n is m, then it is true that whatever holds of n also holds of m
- In other words it could never lead us from true claims to false ones
- This is the essence of a valid inference step

Proof
New Steps

- So we need to think about which inference steps negation, conjunction and disjunction support
  - That is, we need to think about what they mean
  - We've already started doing this:
    1. $\wedge$ takes the 'worst' truth value
    2. $\vee$ takes the 'best' truth value
    3. $\neg$ flips the truth value
- Now we just need to think about what these facts imply in terms of inference steps and proof methods

Conjunction
Elimination

- Suppose you have proved a conjunction $P \wedge Q$
- From looking at the truth table for $\wedge$ or thinking about the meaning of *and*, both P and Q are clearly consequences of $P \wedge Q$
- This inference pattern is called conjunction elimination

Truth Table for $\wedge$

P	Q	$P \wedge Q$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Conjunction Elimination

1. From $P \wedge Q$ you can infer P
2. From $P \wedge Q$ you can infer Q

Conjunction
Elimination

- Conjunction elimination is pretty obvious:
  1. Jay walks and Kay talks
     - So Jay walks
  2. Large(a) $\wedge$ Cube(a)
     - So Cube(a)
- This inference is so obvious that we rarely take the time to mention that we are making it
- In your informal proofs you don't have to mention it either, but you should be aware that you are making it and why it's a valid inference step

Conjunction
Introduction

- There's a similar inference step for inferring a conjunction from its conjuncts:

Conjunction Introduction

If you have proven both P and Q, you can infer $P \land Q$

- Again, this is so obvious that we never mention such a step
- You don't have mention it in your proofs, but you should understand it

Disjunction
Introduction

- Suppose you have proven P
- Then you can infer $P \lor Q$, no matter what Q is
- Why?

Truth Table for $\lor$

P	Q	$P \lor Q$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- If P is true, then $P \lor Q$ is true, regardless of what Q's truth value is!
- This is a valid inference step, since it is guaranteed to lead us from true claims to true claims

Disjunction
Introduction

So $\lor$ supports this inference step:

Disjunction Introduction

1. From A you can infer $A \lor B$
2. From A you can infer $B \lor A$

- It may seem useless, to infer from:
  (1) Likes(jay, circles)
  that
  (2) Likes(jay, circles) $\lor$ Likes(kay, squares)
- But, we will find uses for these kinds of inferences

Informal Proof
A Rule of Thumb

Rule of Thumb for Informal Proofs

In an informal proof, it is always legitimate to move from P to Q if both you & your audience already know that Q is a logical consequence of P

- We've all learned the following equivalences:

DeMorgan's Laws

1. $\neg(P \land Q) \Leftrightarrow \neg P \lor \neg Q$
2. $\neg(P \lor Q) \Leftrightarrow \neg P \land \neg Q$

Double Negation

$\neg\neg P \Leftrightarrow P$

- So in an informal proofs for this class you could say: "From $\neg$(Cube(a) $\land$ Tet(b)) it follows by DeMorgan's Laws that $\neg$Cube(a) $\lor$ $\neg$Tet(b)..."

Informal Proof
A Rule of Thumb

- Of course, if you are asked to prove one of DeMorgan's Laws, or you are talking logic with stranger you shouldn't appeal to DeMorgan's Laws
- So we have three new inference steps, some equivalences and a rule of thumb
- So what?
- Well, now we can prove some stuff

An Example Proof
Argument 1

Argument 1

$$\begin{array}{|l} \neg(A \lor B) \\ \hline \neg A \end{array}$$

- Let's give an informal proof of this inference

Proof of Argument 1

We are given $\neg(A \lor B)$, which is equivalent to $\neg A \land \neg B$ by DeMorgan's Law (2). So $\neg A$ follows immediately.

- We used Conjunction Elimination in this last step, but there was no need to explicitly say so

Another Example Proof
Inference 2

Argument 2

$$\begin{array}{|l} a = b \land \neg\mathsf{Cube}(a) \\ \hline \neg(\mathsf{Cube}(a) \lor \mathsf{Cube}(b)) \end{array}$$

- Let's give an informal proof of this inference too

Proof of Argument 2

We are given that $a = b$ and $\neg\mathsf{Cube}(a)$, so by the indiscernibility of identicals $\neg\mathsf{Cube}(b)$. Now we have $\neg\mathsf{Cube}(a) \land \neg\mathsf{Cube}(b)$, but by DeMorgan's Law (2) this is equivalent to $\neg(\mathsf{Cube}(a) \lor \mathsf{Cube}(b))$, which is our desired conclusion.

Moving On
Proof Methods

- We have discussed:
  1. Conjunction Intro and Elim
  2. Disjunction Intro
- But what about:
  1. Disjunction Elim
  2. Negation!
- As it turns out, these are not formulated as simple rules, but as more structured methods of proof

Proof by Cases
The Basics

- The first method of proof we are going to learn about is called proof by cases
- In short, it will allow us to use disjunctions to prove things
- Let's first look at an example where a disjunction is used to prove something

Proof by Cases
An Example

Disjunctions in a Proof

Apollo went to go buy a smoothie at either the George St. Co-op or Whole Foods. He always buys the cheapest smoothie. Right now, the cheapest smoothie at both the George St. Co-op and Whole Foods is Odwalla. So, if he went to George St. he will buy Odwalla and if he went to Whole Foods he will also buy Odwalla. So either way he'll come back with an Odwalla smoothie.

- Our first premise was a disjunction
- Here's how we used it:
  - We reasoned that if the first disjunct was true, Apollo would come back w/Odwalla
  - We then reasoned that if the second was true, Apollo would come back w/Odwalla
  - We concluded that Apollo would come back w/Odwalla
- Why? Because the meaning of *or* guarantees that at least one disjunct of the first premise is true

Proof by Cases
The Method

So, our strategy was:

- We had $\mathsf{A} \vee \mathsf{B}$ and wanted to show C
- So, we showed that if A was true C was true
- And that if B was true, C was true
- This is a valid proof of C from $\mathsf{A} \vee \mathsf{B}$, since:
  - $\mathsf{A} \vee \mathsf{B}$ guarantees that either A is true or B is true
  - And, we've shown that either way C is true
- This is called proof by cases, since it breaks the proof up into a number of cases, one for each disjunct

Proof by Cases
Official Version

So, more abstractly our strategy was this:

Proof by Cases (Disjunction Elimination)

To prove C from $\mathsf{P}_1 \vee \ldots \vee \mathsf{P}_n$ using this method, show C from each of $\mathsf{P}_1, \ldots \mathsf{P}_n$

- From our disjunctive premise we know at least one disjunct is true
- So showing that the truth of any one of them guarantees the truth to C, suffices to show that C follows from our disjunctive premise

Proof by Cases

An Example

Argument 3

$(\mathsf{Cube}(c) \wedge \mathsf{Small}(c)) \vee (\mathsf{Tet}(c) \wedge \mathsf{Small}(c))$

$\mathsf{Small}(c)$

- Let's give an informal proof of this inference

Proof of Argument 3

We have a disjunctive premise, so we will use the proof by cases method. We have two disjuncts in our premise so we'll break our proof into two cases:

1. Suppose $\mathsf{Cube}(c) \wedge \mathsf{Small}(c)$. Then $\mathsf{Small}(c)$ follows!
2. Suppose $\mathsf{Tet}(c) \wedge \mathsf{Small}(c)$. Then again, $\mathsf{Small}(c)$!

Either way $\mathsf{Small}(c)$ follows

Proof by Cases

Another Example

Claim: the following argument is valid

$\mathsf{Cube}(a) \vee \mathsf{Smaller}(a, b)$

$\neg\mathsf{Cube}(a) \vee \mathsf{Smaller}(a, c)$

$\mathsf{Smaller}(b, c)$

$\mathsf{Smaller}(a, c)$

Proof: Given the first premise, we'll try a proof by cases:

1. Suppose $\mathsf{Cube}(a)$. By the second premise we know that either $\mathsf{Cube}(a)$ is false or $\mathsf{Smaller}(a, c)$. By assumption, $\mathsf{Cube}(a)$ is true. So, it must be the case that $\mathsf{Smaller}(a, c)$
2. Suppose $\mathsf{Smaller}(a, b)$. We are given that $\mathsf{Smaller}(b, c)$ and $\mathsf{Smaller}(,)$ is transitive, so $\mathsf{Smaller}(a, c)$

We've shown that in either case $\mathsf{Smaller}(a, c)$ follows

Proof by Cases

In Class Exercise

Write an informal proof of this argument, phrased in complete, well-formed English sentences. ***Hint***: *try a proof by cases. I encourage you to work in groups.*

$\mathsf{Smaller}(a, c) \vee \mathsf{FrontOf}(a, b)$

$\mathsf{Larger}(a, c) \vee \mathsf{BackOf}(b, a)$

$\mathsf{Between}(c, a, b)$

$\mathsf{FrontOf}(a, b)$

Summary

02.12

- We learned three valid inference steps for the Booleans:
  - Conjunction Intro/Elim and Disjunction Intro
  - You don't have to mention the conjunction steps
- We learned some tricks for informal proofs:
  - You can usually appeal to things like DeMorgan's Laws
  - You can appeal to facts about the vocabulary, e.g. *transitivity*, *inverseness*, at least one disjunct has to be true etc.
- We learned a powerful proof method:
  - Proof by cases