Basics of Quantification

∀ and ∃

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Announcements

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1. Something

Outline

Quantities

In Thought & Talk

- In our daily lives, we think & talk about quantities
  - Some money
  - Every ex-girlfriend
  - Two siblings
  - No friends
  - Many friends
- As it turns out, this thought & talk is governed by interesting logical principles
- These logical principles cannot be captured with the truth-functional connectives

Quantifiers
And Quantifier Phrases

(1) Some money is wasted
(2) Every magician is a vampire
(3) Two cats are meowing
(4) No friends showed up to George's party
(5) Many friends came to my party

- The above sentences contain quantifier phrases
- Simple quantifier phrases have two parts:
  1. A quantifier
  2. A noun
- Last class, we learned how to represent quantifiers and quantifier phrases in FOL

Quantifiers in FOL
Meet ∀ and ∃

- We added the quantifier symbols for FOL:
  The Universal Quantifier $\forall$ (*everything*)
  The Existential Quantifier $\exists$ (*something*)
- And variables
  - FOL has infinitely many variables:
    $\mathsf{t}, \mathsf{u}, \mathsf{v}, \mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{t}_1, \ldots, \mathsf{t}_n, \mathsf{u}_1, \ldots, \mathsf{u}_n, \mathsf{v}_1, \ldots, \mathsf{v}_n, \ldots$
  - They go in the slots of predicates:
    $\mathsf{Cube}(\mathsf{y}), \mathsf{FrontOf}(\mathsf{u}, \mathsf{v}), \mathsf{Between}(\mathsf{z}, \mathsf{u}_{21}, \mathsf{w})$
- Together, these two resources allowed us to represent quantificational sentences

The Universal Quantifier
Universal Statements

- How do you represent a universal statement in FOL?

Example

Everything is small:

$\forall$ x

$\forall\mathsf{x}$ $\mathsf{Small}(\mathsf{x})$

$\forall\mathsf{x}\,\mathsf{Small}(\mathsf{x})$

1. It's a universal statement, so use $\forall$
2. Pick a variable to use, like x
3. Pair $\forall$ with that variable
4. Plug that variable into the predicate of the claim
5. Stick together the two things you've made

- We read $\forall\mathsf{x}\,\mathsf{Small}(\mathsf{x})$ as *For every object x, x is small*
- This is an intuitively correct paraphrase of *Everything is small*

The Quantifier
Existential Statements

- How do you represent a Existential statement in FOL?

Example

Something is small:

$\exists$ x

$\exists\mathsf{x}$ $\mathsf{Small}(\mathsf{x})$

$\exists\mathsf{x}\,\mathsf{Small}(\mathsf{x})$

1. Its a existential statement, so use $\exists$
2. Pick a variable to use, like x
3. Pair $\exists$ with that variable
4. Plug that variable into the predicate of the claim
5. Stick together the two things you've made

- We read $\exists\mathsf{x}\,\mathsf{Small}(\mathsf{x})$ as *For some object x, x is small*
- This is an intuitively correct paraphrase of *Something is small*

Variables

Complete vs. Incomplete

- There's a big difference between these two formulas:
  - (6) Small(x)
  - (7) Small(a)
- (7) makes a claim that is true or false
  - Either a is small or it isn't
- (6) does not
- (6) is an incomplete claim
  - It's like saying *it is small* without telling us what *it* is!
- However, (6) becomes complete when ∃x or ∀x is added
- '∃x Small(x)' **is** either true or false

Variables

Complete vs. Incomplete?

A Question

When exactly does a formula containing variables make a complete claim?

- Does (8) make a complete claim?
  - (8) ∃x (Small(x) ∧ Cube(x))
- What about (9)?
  - (9) ∃x (Small(x) ∧ Cube(x)) ∨ LeftOf(x, a)

The Answer (First Version)

A formula containing variables makes a complete claim just in case every variable appears within the scope of a quantifier symbol attached to that variable

Scope

Some Terminology

Scope

1. A quantificational wff ∀v A is formed by sticking together some wff A and quantifer-phrase ∀v
2. We call A that quantifier's *scope*.

- ∀x (Small(x) ∧ Tet(x))
  - ∀x's Scope: Small(x) ∧ Tet(x)
- ∀x Small(x) ∧ Tet(x)
  - ∀x's Scope: Small(x)

Variables

Complete vs. Incomplete: Revisited

The Answer (First Version)

A formula containing variables makes a complete claim just in case every variable appears within the scope of a quantifier symbol attached to that variable

- Does (8) make a complete claim?
  - (8) ∃x (Small(x) ∧ Cube(x))
    - Yes; both occurrences within scope of ∃x
- What about (9)?
  - (9) ∃x (Small(x) ∧ Cube(x)) ∨ LeftOf(x, a)
    - No; 3rd occurrence outside scope of ∃x

Binding
More Terminology

Bondage

An occurrence of a variable v is bound iff v occurs within the scope of either $\forall v$ or $\exists v$

- 1st & 2nd occurrences of x are bound; 3rd is not

(9) $\exists x\,(\mathsf{Small}(x) \wedge \mathsf{Cube}(x)) \vee \mathsf{LeftOf}(x, a)$

Freedom

An occurrence of a variable v is free iff v does **not** occur within the scope of either $\forall v$ or $\exists v$

- 3rd occurrence of x in (9) is free; 1st & 2nd are not

Two More Things
A New Version of The Answer & Wffs vs. Sentences

The Answer (Second Version)

A formula containing variables makes a complete claim just in case every variable is bound

Sentences vs. Wffs (Approximation)

1. *Well-formed formulas* or *wffs* is the set of **all** grammatical expressions of FOL, including both incomplete claims, like '$\mathsf{Tet}(x)$' and complete ones
2. *Sentences* are formulas that make complete claims; contain no variables or only bound ones

Wffs v. Non-Wffs
Some Examples

Wffs

(10) $\mathsf{Tet}(a)$

(11) $\mathsf{Cube}(y)$

(12) $(\mathsf{Cube}(y) \wedge \mathsf{Tet}(a))$

(13) $(\exists y\,(\mathsf{Cube}(y) \wedge \mathsf{Tet}(a)))$

(14) $(\exists y\,\mathsf{Cube}(y)) \wedge \mathsf{Tet}(a)$

(15) $\mathsf{Tet}(a) \rightarrow (\mathsf{Cube}(b) \wedge \mathsf{Small}(b))$

Non-Wffs

(16) Tet

(17) $(y)\mathsf{Cube}$

(18) $\mathsf{Cube}(y, \mathsf{Small})$

(19) $\wedge\mathsf{Cube}(y)\,\mathsf{Tet}(a)$

(20) $\exists(\mathsf{Cube}(y) \wedge \mathsf{Large}(y))$

(21) $\mathsf{Tet}(a) \rightarrow \mathsf{Cube}(b) \wedge \mathsf{Small}(b)$

- Now that we're clear on the wff v. non-wff distinction, let's draw the one we set out to draw
  - The wff v. sentence distinction

Sentences v. Wffs
Some Examples

Non-Sentence Wffs

(22) $\mathsf{Tet}(y)$

(23) $\neg\mathsf{Cube}(y)$

(24) $(\mathsf{Cube}(y) \wedge \mathsf{Tet}(a))$

(25) $((\exists y\,\mathsf{Cube}(y)) \wedge \mathsf{Tet}(y))$

(26) $(\exists y\,(\mathsf{Cube}(y) \wedge \mathsf{Tet}(x)))$

- Free variables

Sentences

(27) $\mathsf{Tet}(a)$

(28) $\neg\mathsf{Tet}(a)$

(29) $(\mathsf{Cube}(a) \wedge \mathsf{Tet}(a))$

(30) $(\exists y\,(\mathsf{Cube}(y) \wedge \mathsf{Tet}(y)))$

(31) $(\exists y\,(\mathsf{Cube}(y) \wedge (\exists x\,\mathsf{Tet}(x))))$

- No free variables

Semantics & Quantification
Where we Are

- We know what the truth functional connectives ($\wedge, \vee, \neg, \rightarrow, \leftrightarrow$) mean
  - Their meanings are given by their truth tables
  - **Terminology**: *semantics* is the study of **meaning**
- We have not yet learned the semantics for quantifier symbols ($\forall$, $\exists$)
- As it turns out, we cannot provide a semantics for quantifiers using **truth tables**
- Why?

Semantics & Quantification
Why Not Truth Tables

- Truth tables work by explaining the truth of a complex formula in terms of the truth of its parts
  - Example: $\neg$P is T iff P is F
- The problem with using truth tables for quantifiers is that the truth of quantified formulas cannot be determined from the truth of its parts
  - Example: $\forall$x Cube(x) is T iff ???
    - Cube(x) is T? F?
    - Neither!
    - Cube(x) isn't capable of truth or falsity, it's too incomplete!
- So, we can't use truth tables to explain what quantified sentences mean

Satisfaction
The Basic Idea

- If not truth tables, what?
- We'll use a method pioneered by Alfred Tarski (1936)
- He introduced the idea of an object satisfying a formula
- Here's the intuition behind satisfaction
  - Although a formula with a free variable like Cube(x) is neither true nor false, we can think of it being true of some object o
  - Tarski called this special idea of being true of an object *satisfaction*
  - For example, o satisfies Small(x) $\wedge$ Cube(x) iff o is a small cube

Satisfaction
The Precise Definition

Definition of Satisfaction

An object o satisfies a wff S(x) containing x as its only **free variable** iff the following two conditions are met:

1. If we give a o a name that's not taken, call it n_i, then $S(n_i)$ is true
2. $S(n_i)$ is the result of replacing every occurrence of x in S(x) with n_i

- Let's work through some examples in Tarski's World

Domain of Discourse

The Things We're Talking About

- When we ask:
  - Is there an object o that satifies $\mathsf{S}(\mathsf{x})$?
- Which objects should we look at?
- When we communicate, we take as given a collection of objects we're interested in talking about
- Sometimes this collection is absolutely all objects, but more commonly it is some restricted set of objects
- We'll call this set the *domain of discourse*
- So, the answer to our question above is: *the objects in the domain of discourse!*

Domain of Discourse

An Example

Example

- When I say *Every student is sleepy* here and now, which students does it seem most reasonable for me to be talking about?
- You! The students in this classroom (Sadly)
- The domain of discourse is taken to be set of things in this room
- When I say *every student* I restrict your attention to the students in this room

- In Tarski's World the domain of discourse is the collection of blocks on the board

Existential Statements

When are They True?

- Now that we understand satisfaction, we can say when quantified statements are true
- Before we look at the exact definitions, let's get some intuitions clear
- *Something is smelly* is true iff there is some object o and o is smelly
- The truth of $\exists \mathsf{x}\, \mathsf{Smelly}(\mathsf{x})$ can be determined in a similar way:
  - $\exists \mathsf{x}\, \mathsf{Smelly}(\mathsf{x})$ is true iff **some** object o satisfies $\mathsf{Smelly}(\mathsf{x})$
  - That is, if there is some object o such that when you give it an unused name n, $\mathsf{Smelly}(\mathsf{n})$ comes out true
  - If there is no such object, $\exists \mathsf{x}\, \mathsf{Smelly}(\mathsf{x})$ is false

Existential Statements

Official Semantics

Semantics for $\exists$

$\exists \mathsf{x}\, \mathsf{S}(\mathsf{x})$ is TRUE iff there is at least one object that satisfies $\mathsf{S}(\mathsf{x})$

Example

When is $\exists \mathsf{x}\, (\mathsf{Cube}(\mathsf{x}) \wedge \mathsf{Small}(\mathsf{x}))$ true?

- By the semantics for $\exists$:

  (32) Iff there is at least one object that satisfies $\mathsf{Cube}(\mathsf{x}) \wedge \mathsf{Small}(\mathsf{x})$
- By the definition of satisfaction (33) amounts to:
  - Iff when we give o some unused name n, $\mathsf{Cube}(\mathsf{n}) \wedge \mathsf{Small}(\mathsf{n})$ comes out true

Existential Statements

Examples

- The way to understand these definitions is by going through examples
- Let's go to Tarski's World and evaluate some existential claims

Universal Statements

When are They True?

- When are universal statements are true?
- Before we look at the exact definition, let's get some intuitions clear
- *Everything is beautiful* is true iff for every object o, o is smelly
- The truth of $\forall x\, \mathsf{Beautiful}(x)$ can be determined in a similar way:
  - Consider whether every object o in the domain of discourse satisfies $\mathsf{Beautiful}(x)$
  - That is, for every object o see whether when you give it an unused name n, $\mathsf{Beautiful}(\mathsf{n})$ comes out true
  - If so, then $\forall x\, \mathsf{Beautiful}(x)$ is true
  - Otherwise, it is false
- Okay, let's see the precise definition

Universal Statements

Official Semantics

Semantics for $\forall$

$\forall x\, \mathsf{S}(x)$ is TRUE iff every object satisfies $\mathsf{S}(x)$

Example

When is $\forall x\, (\mathsf{Cube}(x) \wedge \mathsf{Small}(x))$ true?

- By the semantics for $\forall$:

  (33) Iff every object o satisfies $\mathsf{Cube}(x) \wedge \mathsf{Small}(x)$
- By the definition of satisfaction (33) amounts to:
  - Iff when we give each o some unused name n, $\mathsf{Cube}(\mathsf{n}) \wedge \mathsf{Small}(\mathsf{n})$ comes out true

Universal Statements

Examples

- The way to understand these definitions is by going through examples
- Let's go to Tarski's World and evaluate some universal claims