Informal Proofs Formal Proofs	Informal Proofs Formal Proofs Announcements For 06.11.08
The Logic of Conditionals Informal & Formal Proofs	<ul> <li>HW6 is due now</li> <li>The midterm is a week from today!</li> <li>The practice midterm/HW7 is available now</li> </ul>
William Starr	<ul> <li>Class this Thursday (03.05) will be a review</li> <li>We will work through select problems from the practice midterm</li> </ul>
03.03.09	• So bring your questions
William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University 1/30	William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University 2/30
Informal Proofs Formal Proofs Outline	Informal Proofs Formal Proofs Material Conditional Modus Ponens
1 Informal Proofs	Truth Table for $\rightarrow$ Modus Ponens $P   Q     P \rightarrow Q$ If you have established $P \rightarrow Q$ T   T   TT
2 Formal Proofs	T   T   T     T   F   F     F   T     T   T     •   This rule is also known as

- conditional elimination
- Why is modus ponens correct?

Т

F F

- $\bullet~{\rm If}~P\to Q~{\rm is}~\tau$  and  $P~{\rm is}~\tau,$  then  $Q~{\rm must}$  be  $\tau$
- $\bullet\,$  So when you have  $\mathsf{P}\to\mathsf{Q}$  and  $\mathsf{P},$  you have  $\mathsf{Q}!$

# Material Conditional

Modus Ponens at Work

#### A Simple Application of Modus Ponens

Suppose you are told that if a is a cube, then it is small, and that a is indeed a cube. Then it follows by modus ponens that a is small. Symbolically:

 $\mathsf{Cube}(\mathsf{a}) \to \mathsf{Small}(\mathsf{a}) \ \mathrm{and} \ \mathsf{Cube}(\mathsf{a}), \ \mathrm{therefore} \ \mathsf{Small}(\mathsf{a}).$ 

#### Modus Ponens Again

Suppose you are told that if a is either a cube or a tetrahedron, then a is in the same row as b, and that a is a cube. Then it follows that a is a cube or a tetrahedron. So by modus ponens, it follows that a is in the same row as b. Symbolically:

We are given that  $(Cube(a) \lor Tet(a)) \rightarrow SameRow(a, b)$  and Cube(a). By the second claim:  $Cube(a) \lor Tet(a)$  follows. Then by modus ponens it follows that SameRow(a, b).

William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University

7/30

#### nformal Proofs Formal Proofs

Conditional Proof An Example

Let's use conditional proof and modus ponens to give a proof of:

Argument 1

$$\begin{array}{l} {\sf Tet}({\sf a}) \to {\sf Tet}({\sf b}) \\ \\ {\sf Tet}({\sf b}) \to {\sf Tet}({\sf c}) \\ \\ \hline \\ {\sf Tet}({\sf a}) \to {\sf Tet}({\sf c}) \end{array}$$

Our goal is a conditional, so we use conditional proof.

**Proof**: Suppose Tet(a). Then by premise 1 Tet(b) follows by modus ponens. But then we may now again use modus ponens and premise 2 to infer Tet(c). This is the consequent of our goal, so we have successfully completed our conditional proof.

#### Informal Proofs Formal Proofs

# Conditional Proof

## The Method of Conditional Proof

To prove  $P \to Q$ , temporarily assume P. If you can show Q with this additional assumption, you can infer  $P \to Q$ 

Truth Table for $ ightarrow$		
Ρ	Q	$P\toQ$
Т	Т	Т
Т	F	F
F	Т	Т
F	F	Т

- The only way for  $P \rightarrow Q$  to be F is for P to be true and Q be F
- So, if you can show that when P is T Q is also T, you've shown that P → Q cannot be false, i.e. that it is true!

William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University



Let's do  $\mathbf{exercise}~\mathbf{8.4}$  on the chalkboard

8.4 | The unicorn, if horned, is elusive and dangerous.

If elusive or mythical, the unicorn is rare.

If a mammal, the unicorn is not rare.

The unicorn, if horned, is not a mammal.

Give an informal proof of the validity of this argument, using conditional proof.

# The Material Biconditional Elimination

$Truth Table for \leftrightarrow$		
Ρ	Q	$P \leftrightarrow Q$
Т	Т	Т
Т	F	F
F	Т	F
F	F	Т
	_	

# Biconditional Elimination

If you have established either  $P \leftrightarrow Q$  or  $Q \leftrightarrow P$  and P, then you can infer Q.

• This rule is also known as biconditional elimination

- Why is this correct?
  - $\bullet~\mbox{If}~\mathsf{P}\leftrightarrow\mathsf{Q}~\mbox{is}~\tau$  and  $\mathsf{P}~\mbox{is}~\tau,$  then  $\mathsf{Q}~\mbox{must}~\mbox{be}~\tau$
  - $\bullet\,$  Similarly, if  $\mathsf{P}\leftrightarrow\mathsf{Q}$  is  $\tau$  and  $\mathsf{Q}$  is  $\tau,$  then  $\mathsf{P}$  is  $\tau$

William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University

12/30

#### nformal Proofs Formal Proofs

Proving Biconditionals Conditional Proof Twice Over

#### How to Prove a Biconditional

To prove  $\mathsf{P} \leftrightarrow \mathsf{Q}$ , first, use conditional proof to prove  $\mathsf{P} \to \mathsf{Q}$ . Then use conditional proof again to prove  $\mathsf{Q} \to \mathsf{P}$ . Showing these two conditionals suffices to prove the biconditional.

- How do you prove a biconditional like  $\mathsf{P} \leftrightarrow \mathsf{Q}$ ?
- We know that  $P \rightarrow Q$  is equivalent to  $(P \rightarrow Q) \land (Q \rightarrow P)$
- But we know how to prove  $(P \rightarrow Q) \land (Q \rightarrow P)$ :
  - $\bullet~$  Use conditional proof to show  $\mathsf{P}\to\mathsf{Q}$
  - $\bullet\,$  Then use conditional proof to show  $\mathsf{Q}\to\mathsf{P}\,$

#### Informal Proofs Formal Proofs

# The Material Biconditional Elimination

#### **Biconditional Elimination Example**

Suppose you are told that a is in the same column as b if and only if a is a tetrahedron, and that a is tetrahedron. Then by biconditional elimination, it follows that a is in the same column as b. Symbolically:

 $\mathsf{SameCol}(\mathsf{a},\mathsf{b}) \leftrightarrow \mathsf{Tet}(\mathsf{a}) \text{ and } \mathsf{Tet}(\mathsf{a}), \text{ so } \mathsf{SameCol}(\mathsf{a},\mathsf{b}).$ 

#### William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University

#### 13/30

# Informal Proofs Formal Proofs Proving A Biconditional

#### <u>An</u> Example

Let's give an informal proof of this argument:

 $\begin{tabular}{c} Cube(a) \leftrightarrow Cube(b) \\ \hline Cube(b) \leftrightarrow Cube(c) \\ \hline Cube(a) \leftrightarrow Cube(c) \\ \hline \end{tabular}$ 

Our goal is a biconditional, so we do two conditional proofs.

#### **Proof**:

- Isrst we'll show Cube(a) → Cube(c) by conditional proof. Suppose Cube(a). Then from premise 1 Cube(b) follows by biconditional elimination. From this and premise 2 it follows by biconditional elimination again that Cube(c). So, Cube(a) → Cube(c)
- Now we'll show  $Cube(c) \rightarrow Cube(a)$  by conditional proof. Suppose Cube(c). Then from premise 2 Cube(b) follows by biconditional elimination. From this and premise 1 it follows by biconditional elimination again that Cube(a). So,  $Cube(c) \rightarrow Cube(a)$ .

By these two conditional proofs, it follows that  $\mathsf{Cube}(\mathsf{a}) \leftrightarrow \mathsf{Cube}(\mathsf{c})$ 

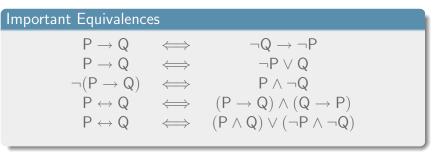
#### In Class Exercise

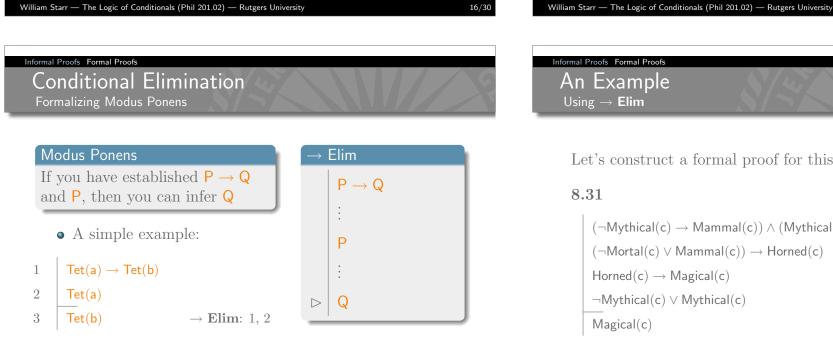
Exercise 8.5: Construct an informal proof of the argument. Here's the argument translated into FOL.

```
(Horned(u) \rightarrow (Elusive(u) \land Magical(u)))
      \wedge (\neg Horned(u) \rightarrow (\neg Elusive(u) \land \neg Magical(u)))
\negHorned(u) \rightarrow \negMythical(u)
Horned(u) \leftrightarrow (Magical(u) \lor Mythical(u))
```

nformal Proofs Formal Proofs Conditionals Additional Steps

Some additional equivalences that are useful for informal proofs:





 $\bullet \rightarrow \text{Elim}$  is the formal counterpart to our informal rule called modus ponens

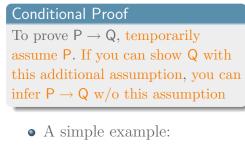


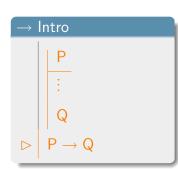


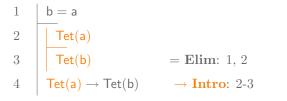
Let's construct a formal proof for this argument:

```
(\neg Mythical(c) \rightarrow Mammal(c)) \land (Mythical(c) \rightarrow \neg Mortal(c))
(\neg Mortal(c) \lor Mammal(c)) \rightarrow Horned(c)
Horned(c) \rightarrow Magical(c)
\negMythical(c) \lor Mythical(c)
```

# Conditional Introduction Formalizing Conditional Proof





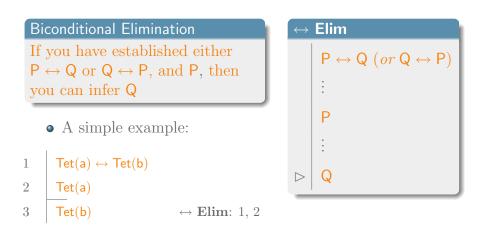


William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University

23/30

26/30





 ↔ Elim is the formal counterpart to our informal rule called biconditional elimination

#### Informal Proofs Formal Proofs

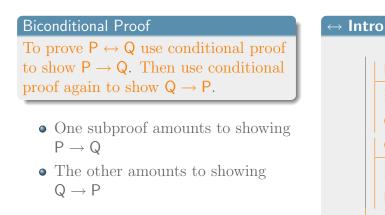
# Conditional Rules Another Example with $\rightarrow$ Elim & $\rightarrow$ Intro

Let's do exercise **8.32**. This involves a formal version of the informal proof we did for exercise **8.4**. We will use the informal proof to guide us.



Informal Proofs Formal Proofs

↔ Intro Formalizing Biconditional Proof



Q

Q

Ρ

 $\mathsf{P} \leftrightarrow \mathsf{Q}$ 

24/30



Let's do a proof in Fitch for a simple example that uses both  $\leftrightarrow$  **Intro** and  $\leftrightarrow$  **Elim**:

#### 8.25 Transitivity of the Biconditional

$$\begin{array}{c|c} A \leftrightarrow B \\ \hline B \leftrightarrow C \\ \hline A \leftrightarrow C \end{array}$$

Informal Proofs Formal Proofs  $\leftrightarrow$  Elim &  $\leftrightarrow$  Intro In Class Exercise

You constructed an informal proof for this argument, now turn this into a formal proof:

**Hint**: You should do two subproofs and then apply  $\leftrightarrow$  **Intro** to get the conclusion

1 In the first subproof, assume Horned(c), show  $Magical(c) \lor Mythical(c)$ 

In the second one, assume Magical(c) ∨ Mythical(c), show Horned(c). It may be easier to show Horned(c) using indirect proof (assume ¬Horned(c) and derive ⊥)

William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University

29/30

William Starr — The Logic of Conditionals (Phil 201.02) — Rutgers University