The Logic of Conditionals

Informal & Formal Proofs

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Announcements

For 06.11.08

1. HW6 is due now
2. The midterm is a week from today!
3. The practice midterm/HW7 is available now
4. Class this Thursday (03.05) will be a review
   - We will work through select problems from the practice midterm
   - So bring your questions

Outline

Material Conditional

Modus Ponens

Truth Table for $\rightarrow$

P	Q	P $\rightarrow$ Q
T	T	T
T	F	F
F	T	T
F	F	T

Modus Ponens

If you have established P $\rightarrow$ Q and P, then you can infer Q

- This rule is also known as conditional elimination

- Why is modus ponens correct?
   - If P $\rightarrow$ Q is T and P is T, then Q **must** be T
   - So when you have P $\rightarrow$ Q and P, you have Q!

Material Conditional

Modus Ponens at Work

A Simple Application of Modus Ponens

Suppose you are told that if a is a cube, then it is small, and that a is indeed a cube. Then it follows by modus ponens that a is small. Symbolically:

$\mathsf{Cube(a)} \rightarrow \mathsf{Small(a)}$ and $\mathsf{Cube(a)}$, therefore $\mathsf{Small(a)}$.

Modus Ponens Again

Suppose you are told that if a is either a cube or a tetrahedron, then a is in the same row as b, and that a is a cube. Then it follows that a is a cube or a tetrahedron. So by modus ponens, it follows that a is in the same row as b. Symbolically:

We are given that $(\mathsf{Cube(a)} \vee \mathsf{Tet(a)}) \rightarrow \mathsf{SameRow(a, b)}$ and $\mathsf{Cube(a)}$. By the second claim: $\mathsf{Cube(a)} \vee \mathsf{Tet(a)}$ follows. Then by modus ponens it follows that $\mathsf{SameRow(a, b)}$.

Conditional Proof

The Method

The Method of Conditional Proof

To prove $\mathsf{P} \rightarrow \mathsf{Q}$, temporarily assume P. If you can show Q with this additional assumption, you can infer $\mathsf{P} \rightarrow \mathsf{Q}$

Truth Table for $\rightarrow$

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

- The only way for $\mathsf{P} \rightarrow \mathsf{Q}$ to be F is for P to be true and Q be F
- So, if you can show that when P is T Q is also T, you've shown that $\mathsf{P} \rightarrow \mathsf{Q}$ cannot be false, i.e. that it is true!

Conditional Proof

An Example

Let's use conditional proof and modus ponens to give a proof of:

Argument 1

$\mathsf{Tet(a)} \rightarrow \mathsf{Tet(b)}$
$\mathsf{Tet(b)} \rightarrow \mathsf{Tet(c)}$
———
$\mathsf{Tet(a)} \rightarrow \mathsf{Tet(c)}$

Our goal is a conditional, so we use conditional proof.

Proof: Suppose $\mathsf{Tet(a)}$. Then by premise 1 $\mathsf{Tet(b)}$ follows by modus ponens. But then we may now again use modus ponens and premise 2 to infer $\mathsf{Tet(c)}$. This is the consequent of our goal, so we have successfully completed our conditional proof.

Conditional Proof

Another Example

Let's do **exercise 8.4** on the chalkboard

8.4

The unicorn, if horned, is elusive and dangerous.
If elusive or mythical, the unicorn is rare.
If a mammal, the unicorn is not rare.
———
The unicorn, if horned, is not a mammal.

Give an informal proof of the validity of this argument, using conditional proof.

The Material Biconditional
Elimination

Truth Table for ↔

P	Q	P ↔ Q
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Elimination

If you have established either $\mathsf{P} \leftrightarrow \mathsf{Q}$ or $\mathsf{Q} \leftrightarrow \mathsf{P}$ and P, then you can infer Q.

- This rule is also known as biconditional elimination

- Why is this correct?
  - If $\mathsf{P} \leftrightarrow \mathsf{Q}$ is T and P is T, then Q **must** be T
  - Similarly, if $\mathsf{P} \leftrightarrow \mathsf{Q}$ is T and Q is T, then P is T

The Material Biconditional
Elimination

Biconditional Elimination Example

Suppose you are told that a is in the same column as b if and only if a is a tetrahedron, and that a is tetrahedron. Then by biconditional elimination, it follows that a is in the same column as b. Symbolically:

$\mathsf{SameCol(a, b)} \leftrightarrow \mathsf{Tet(a)}$ and $\mathsf{Tet(a)}$, so $\mathsf{SameCol(a, b)}$.

Proving Biconditionals
Conditional Proof Twice Over

How to Prove a Biconditional

To prove $\mathsf{P} \leftrightarrow \mathsf{Q}$, first, use conditional proof to prove $\mathsf{P} \rightarrow \mathsf{Q}$. Then use conditional proof again to prove $\mathsf{Q} \rightarrow \mathsf{P}$. Showing these two conditionals suffices to prove the biconditional.

- How do you prove a biconditional like $\mathsf{P} \leftrightarrow \mathsf{Q}$?
- We know that $\mathsf{P} \rightarrow \mathsf{Q}$ is equivalent to $(\mathsf{P} \rightarrow \mathsf{Q}) \wedge (\mathsf{Q} \rightarrow \mathsf{P})$
- But we know how to prove $(\mathsf{P} \rightarrow \mathsf{Q}) \wedge (\mathsf{Q} \rightarrow \mathsf{P})$:
  - Use conditional proof to show $\mathsf{P} \rightarrow \mathsf{Q}$
  - Then use conditional proof to show $\mathsf{Q} \rightarrow \mathsf{P}$

Proving A Biconditional
An Example

Let's give an informal proof of this argument:

$$\begin{array}{|l} \mathsf{Cube(a)} \leftrightarrow \mathsf{Cube(b)} \\ \mathsf{Cube(b)} \leftrightarrow \mathsf{Cube(c)} \\ \hline \mathsf{Cube(a)} \leftrightarrow \mathsf{Cube(c)} \end{array}$$

Our goal is a biconditional, so we do two conditional proofs.

Proof:

1. First we'll show $\mathsf{Cube(a)} \rightarrow \mathsf{Cube(c)}$ by conditional proof. Suppose $\mathsf{Cube(a)}$. Then from premise 1 $\mathsf{Cube(b)}$ follows by biconditional elimination. From this and premise 2 it follows by biconditional elimination again that $\mathsf{Cube(c)}$. So, $\mathsf{Cube(a)} \rightarrow \mathsf{Cube(c)}$
2. Now we'll show $\mathsf{Cube(c)} \rightarrow \mathsf{Cube(a)}$ by conditional proof. Suppose $\mathsf{Cube(c)}$. Then from premise 2 $\mathsf{Cube(b)}$ follows by biconditional elimination. From this and premise 1 it follows by biconditional elimination again that $\mathsf{Cube(a)}$. So, $\mathsf{Cube(c)} \rightarrow \mathsf{Cube(a)}$.

By these two conditional proofs, it follows that $\mathsf{Cube(a)} \leftrightarrow \mathsf{Cube(c)}$

Proving a Biconditional
In Class Exercise

Exercise **8.5**: Construct an informal proof of the argument. Here's the argument translated into FOL.

$$\begin{array}{|l} (\mathsf{Horned}(u) \rightarrow (\mathsf{Elusive}(u) \wedge \mathsf{Magical}(u))) \\ \quad \wedge (\neg \mathsf{Horned}(u) \rightarrow (\neg \mathsf{Elusive}(u) \wedge \neg \mathsf{Magical}(u))) \\ \neg \mathsf{Horned}(u) \rightarrow \neg \mathsf{Mythical}(u) \\ \hline \mathsf{Horned}(u) \leftrightarrow (\mathsf{Magical}(u) \vee \mathsf{Mythical}(u)) \end{array}$$

Conditionals
Additional Steps

Some additional equivalences that are useful for informal proofs:

Important Equivalences

$$\begin{array}{ccc} P \rightarrow Q & \Longleftrightarrow & \neg Q \rightarrow \neg P \\ P \rightarrow Q & \Longleftrightarrow & \neg P \vee Q \\ \neg(P \rightarrow Q) & \Longleftrightarrow & P \wedge \neg Q \\ P \leftrightarrow Q & \Longleftrightarrow & (P \rightarrow Q) \wedge (Q \rightarrow P) \\ P \leftrightarrow Q & \Longleftrightarrow & (P \wedge Q) \vee (\neg P \wedge \neg Q) \end{array}$$

Conditional Elimination
Formalizing Modus Ponens

Modus Ponens

If you have established $P \rightarrow Q$ and P, then you can infer Q

- A simple example:

$$\begin{array}{r|ll} 1 & \mathsf{Tet}(a) \rightarrow \mathsf{Tet}(b) & \\ 2 & \mathsf{Tet}(a) & \\ \hline 3 & \mathsf{Tet}(b) & \rightarrow \textbf{Elim}: 1, 2 \end{array}$$

$\rightarrow$ Elim

$$\begin{array}{l|l} & P \rightarrow Q \\ & \vdots \\ & P \\ & \vdots \\ \triangleright & Q \end{array}$$

- $\rightarrow$ **Elim** is the formal counterpart to our informal rule called **modus ponens**

An Example
Using $\rightarrow$ **Elim**

Let's construct a formal proof for this argument:

8.31

$$\begin{array}{|l} (\neg \mathsf{Mythical}(c) \rightarrow \mathsf{Mammal}(c)) \wedge (\mathsf{Mythical}(c) \rightarrow \neg \mathsf{Mortal}(c)) \\ (\neg \mathsf{Mortal}(c) \vee \mathsf{Mammal}(c)) \rightarrow \mathsf{Horned}(c) \\ \mathsf{Horned}(c) \rightarrow \mathsf{Magical}(c) \\ \neg \mathsf{Mythical}(c) \vee \mathsf{Mythical}(c) \\ \hline \mathsf{Magical}(c) \end{array}$$

Conditional Introduction
Formalizing Conditional Proof

Conditional Proof

To prove $P \to Q$, temporarily assume P. If you can show Q with this additional assumption, you can infer $P \to Q$ w/o this assumption

$\to$ Intro

	$\quad$ | P
	$\quad$ | $\vdots$
	$\quad$ | Q
$\triangleright$	$P \to Q$

- A simple example:

1	$b = a$	
2	$\quad$ $Tet(a)$	
3	$\quad$ $Tet(b)$	$=$ **Elim**: 1, 2
4	$Tet(a) \to Tet(b)$	$\to$ **Intro**: 2-3

Conditional Rules
Another Example with $\to$ **Elim** & $\to$ **Intro**

Let's do exercise **8.32**. This involves a formal version of the informal proof we did for exercise **8.4**. We will use the informal proof to guide us.

$\leftrightarrow$ Elim
Formalizing Biconditional Elimination

Biconditional Elimination

If you have established either $P \leftrightarrow Q$ or $Q \leftrightarrow P$, and P, then you can infer Q

$\leftrightarrow$ Elim

	$P \leftrightarrow Q$ (*or* $Q \leftrightarrow P$)
	$\vdots$
	P
	$\vdots$
$\triangleright$	Q

- A simple example:

1	$Tet(a) \leftrightarrow Tet(b)$	
2	$Tet(a)$	
3	$Tet(b)$	$\leftrightarrow$ **Elim**: 1, 2

- $\leftrightarrow$ **Elim** is the formal counterpart to our informal rule called **biconditional elimination**

$\leftrightarrow$ Intro
Formalizing Biconditional Proof

Biconditional Proof

To prove $P \leftrightarrow Q$ use conditional proof to show $P \to Q$. Then use conditional proof again to show $Q \to P$.

- One subproof amounts to showing $P \to Q$
- The other amounts to showing $Q \to P$

$\leftrightarrow$ Intro

	$\quad$ | P
	$\quad$ | $\vdots$
	$\quad$ | Q
	$\quad$ | Q
	$\quad$ | $\vdots$
	$\quad$ | P
$\triangleright$	$P \leftrightarrow Q$

↔ Elim & ↔ Intro

A Simple Example

Let's do a proof in Fitch for a simple example that uses both ↔ **Intro** and ↔ **Elim**:

8.25 *Transitivity of the Biconditional*

$$\begin{array}{|l} A \leftrightarrow B \\ B \leftrightarrow C \\ \hline A \leftrightarrow C \end{array}$$

↔ Elim & ↔ Intro

In Class Exercise

You constructed an informal proof for this argument, now turn this into a formal proof:

8.33

$$\begin{array}{|l} (\mathsf{Horned}(c) \rightarrow (\mathsf{Elusive}(c) \wedge \mathsf{Magical}(c))) \\ \quad \wedge (\neg\mathsf{Horned}(c) \rightarrow (\neg\mathsf{Elusive}(c) \wedge \neg\mathsf{Magical}(c))) \\ \neg\mathsf{Horned}(c) \rightarrow \neg\mathsf{Mythical}(c) \\ \hline \mathsf{Horned}(c) \leftrightarrow (\mathsf{Magical}(c) \vee \mathsf{Mythical}(c)) \end{array}$$

Hint: You should do two subproofs and then apply ↔ **Intro** to get the conclusion

1. In the first subproof, assume Horned(c), show Magical(c) ∨ Mythical(c)
2. In the second one, assume Magical(c) ∨ Mythical(c), show Horned(c). It may be easier to show Horned(c) using indirect proof (assume ¬Horned(c) and derive ⊥)