The Logic of Boolean Connectives

Tautological Equivalence & Consequence

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Announcements

For 02.10

1. HW3 is due now

Outline

Review

The Boolean Connectives

Truth Table for ¬

P	¬P
TRUE	FALSE
FALSE	TRUE

Truth Table for ∧

P	Q	P ∧ Q
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Truth Table for ∨

P	Q	P ∨ Q
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- ¬ flips the value
- ∧ takes the ‘worst’ value
- ∨ takes the ‘best’ value

The Basics
Step 1: The Reference Columns

- We are going to construct a truth table for

  (1) Cube(a) ∨ ¬Cube(a)

First, some columns:

1. A column for each atomic sub-sentence of (1), called a reference column
2. A column for (1) itself

Truth Table for (1)

Cube(a)	Cube(a) ∨ ¬Cube(a)
T	
F	

- Now, we fill the reference column w/truth values
- One row for each unique logical possibility

The Basics
Step 2: Inside Out

Truth Table for (1)

Cube(a)	Cube(a) ∨ ¬Cube(a)
T	F
F	T

- Next, we fill in the column beneath the innermost connective ¬:
  - In the first row, Cube(a) is T so ¬Cube(a) is F
  - In the second row, Cube(a) is F so ¬Cube(a) is T

The Basics
Step 3: The Main Connective

Truth Table for (1)

Cube(a)	Cube(a) ∨ ¬Cube(a)
T	T F
F	T T

- Lastly, we fill in the columns beneath the outermost connective ∨:
  - In the first row, Cube(a) is T and ¬Cube(a) is F, so their disjunction is T
  - In the first row, Cube(a) is F and ¬Cube(a) is T, so their disjunction is T
- This column lists every logically possible truth value for (1)

The Basics
Summary

How to Construct a Truth Table for P

1. **Reference Columns**: Draw a column for each atomic sub-sentence of P, these columns are called the reference columns and are filled with every possible combination of truth-values for the sub-sentences
2. **Inside Out**: Draw a column for P itself. Then fill in the column below P's innermost connective. Repeat for the next innermost connective, until you get to the main connective.
3. **Main Connective**: Fill in the column under the main connective. This row lists the possible truth values of P

Equivalence

Two Varieties

Logical Equivalence

Two sentences are logically equivalent if and only if they have the same truth values in every possible situation

- For example: Tet(a) and ¬¬Tet(a) are logically equivalent

Tautological Equivalence

Two sentences are tautologically equivalent just in case the columns under their main connectives in a joint truth table are identical

- What is a joint truth table?

Equivalence

Joint Truth Tables

- The idea of a joint truth table is quite simple, just add a column on the right for another formula and calculate as before

Joint Truth Table

P	Q	¬	(P ∧ Q)	¬P	∨	¬Q
T	T	F	T	F	F	F
T	F	T	F	F	T	T
F	T	T	F	T	T	F
F	F	T	F	T	T	T

1. Ref. columns
2. Inner connectives
3. Main connectives

- The columns under the main connectives are identical
- So, these two sentences are tautologically equivalent

Equivalence

Logical vs. Tautological

- We started by characterizing two kinds of equivalence
  1. Logical
  2. Tautological
- Every two sentences that are tautologically equivalent are logically equivalent
- Does the reverse hold?
- No, this pair is logically equivalent:

  (2) $a = b \land \text{Cube}(a)$

  (3) $a = b \land \text{Cube}(b)$
- But we can show with Boole that they aren't tautologically equivalent

Consequence

Two Varieties Again

Logical Consequence

C is a logical consequence of $P_1, \ldots, P_n$ just in case it is logically impossible for C to be false while $P_1, \ldots, P_n$ are true

- We've already met this concept of validity/consequence
- It doesn't help us much with figuring out whether an argument is valid
- Proof provides one method, truth tables another:

Tautological Consequence

C is a tautological consequence of $P_1, \ldots P_n$ just in case every row in their **joint truth table** that lists T under $P_1, \ldots, P_n$ also lists T under C

Tautological Consequence

An Example

Argument 1

$$\begin{array}{|l} A \lor B \\ \neg A \\ \hline B \end{array}$$

Joint Truth Table for Argument 1

A	B	A ∨ B	¬A	B
T	T	T	F	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

- First two columns for the premises
- Last column for conclusion
- Every row where both premises are T, the conclusion is T
- So B is a tautological consequence of $A \lor B$ and $\neg A$

Tautological Consequence

Another Example: Exercise 4.20

Let's run through exercise 4.20

Tautological Consequence

In-Class Exercise

Exercise **4.21**

$$\begin{array}{|l} \text{Taller(claire, max)} \lor \text{Taller(max, claire)} \\ \text{Taller(claire, max)} \\ \hline \neg\text{Taller(max, claire)} \end{array}$$

Consequence

Questions

1. If C is a tautological consequence of $P_1, \ldots, P_n$, is C a logical consequence of $P_1, \ldots P_n$?
   - Yes, clearly
2. If C is a logical consequence of $P_1, \ldots, P_n$, is C a tautological consequence of $P_1, \ldots P_n$?
   - No, lets show it using Boole to construct a joint truth table for this argument:

Argument 2

$$\begin{array}{|l} a = b \land b = c \\ \hline a = c \end{array}$$

Taut Con

Tautological Consequence in Fitch

- Truth tables provide a powerful but purely mechanical procedure to test for logical consequence
- But, they often get really tedious and long
- But, that's what computers are good at
- Fitch has a built-in mechanism for testing for tautological consequence
  - **Taut Con**
- Much like **Ana Con**, this is not a rule of inference, but a computational mechanism
- Let's run through exercise 4.26

Summary

What we did today

- We reviewed the construction of truth tables, by hand & with Boole
- We applied truth tables to the problem of precisely defining:
  1. Logical equivalence
  2. Logical consequence
- We came up with two similar concepts:
  1. Tautological equivalence
  2. Tautological consequence
- In each case Tautological implied Logical, but Logical did not imply Tautological