Modality and Modal Logic

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1 Introduction

- Big picture:
  - Explain why some claims follows from others
  - Use a formal language with a semantics:
    - ▶ Represent sentence used to make claim in a formal language
    - ▶ Give the formal language a semantics
    - ▶ Define semantic consequence ($\vDash$) relation for formal language
    - ▶ See whether it captures which claims follow from which
- Propositional logic:
  - **Syntax**:
    - ▶ Atomic formulas: $\mathsf{P}, \mathsf{Q}, \mathsf{R}, \ldots$, Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
  - **Semantics**:
    - ▶ *Valuation*: v maps every atomic to a truth-value, 1 or 0
    - ▶ *Truth (wrt v) Definition*:

      (1) $[\![\mathsf{A}]\!]_v = 1 \iff v(\mathsf{A}) = 1$

      (2) $[\![\neg\phi]\!]_v = 1 \iff [\![\phi]\!]_v = 0$

      (3) $[\![\phi \rightarrow \psi]\!]_v = 1 \iff [\![\phi]\!]_v = 0$ or $[\![\psi]\!]_v = 1$

      ⋮
  - **Consequence**:
    - ▶ $\phi_1, \ldots, \phi_n \vDash \psi \iff$ for every v: $[\![\psi]\!]_v = 1$ if $[\![\phi_1]\!]_v = \cdots = [\![\phi_n]\!]_v = 1$
- The important points:
  - Truth-value of complex formulas are determined by the truth-values of their parts
    - ▶ E.g. $\neg\phi$'s truth-value is determined by ϕ's
  - Truth is relative to an assignment:
    - ▶ Only truth wrt a valuation v

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- ▶ A valuation says how the 'atomic facts' come out
- ▶ It suffices to determine the truth value of every formula
- Consequence is defined in terms of truth

1.1 Modal Operators

- Consider *could*:

  (1) Will could eat a bottle of glue (True)
- Is truth value of (1), determined by truth value of:
  - *Will ate a bottle of glue* (False)
- Can't be:
  - *Will ate the Earth* (False)
  - *Would could eat the Earth* (False)
- **The Point**:
  - Truth of modal sentences are not determined by the truth of their parts
  - It isn't possible to capture truth-conditions of modal claims in propositional logic!

1.2 Conditionals

- Indicative vs. Subjunctive:
  - Indicative:

    (2) If Oswald didn't shoot Kennedy, someone else did
  - Subjunctive:

    (3) If Oswald hadn't shot Kennedy, someone else would have
- Counterfactual:
  - Subjunctive conditional w/false antecedent

    (4) If I were 7 feet tall, I would be taller than the Eiffel Tower
- Many problems for propositional logic analysis!
- **Problem 1** (Counterfactuals):
  - (4) is clearly false, but the corresponding material conditional is true!
- **Problem 2** (Logic):
  - The material conditional validates the following

    Material Negation $\neg(\mathsf{P} \rightarrow \mathsf{Q}) \vDash \mathsf{P}$

    Antecedent Negation $\neg\mathsf{P} \vDash \mathsf{P} \rightarrow \mathsf{Q}$

Contraposition $\mathsf{P} \to \mathsf{Q} \vDash \neg\mathsf{Q} \to \neg\mathsf{P}$

- But the first two are absurd and the last seems to have counterexamples:
  (5) It is not true that if God exists, he is made of spagetti. So God exists
  (6) Bob didn't dance, so if Bob danced, he's a turnip.
  (7) If it rains, it won't pour. So, if it pours it won't rain. (Adams 1975: 15)
- Other problems pointed out by Lewis (1914: 243-4)

2 Modal Logic

- **Goal**: a better analysis of conditionals and modal operators
- Propositional modal logic:
  - **Syntax**:
    - Atomic formulas: $\mathsf{P}, \mathsf{Q}, \mathsf{R}, \ldots$, Connectives: $\neg, \wedge, \vee, \to, \leftrightarrow, \Box, \Diamond$
      - Strict Conditional: $\Box(\phi \to \psi)$
  - **Kripke Semantics**:
    - *Model*: $\mathcal{M} = \langle\langle W, R\rangle, v\rangle$
      - Frame: $\langle W, R\rangle$
    - *Set of Possible Worlds*: W, different complete ways the world could be
    - *Accessibility*: R, a set of pairs of worlds
      - R catalogs which worlds are possible with respect to each-other
      - $R(w, w')$ means that w' is possible wrt to w
    - *Valuation*: v maps every atomic and world to a truth-value at that world, 1 or 0
    - *Truth (wrt v, w, R) Definition*:
      (1) $[\![\mathsf{A}]\!]^R_{v,w} = 1 \iff v(\mathsf{A}, w) = 1$
      (2) $[\![\neg\phi]\!]^R_{v,w} = 1 \iff [\![\phi]\!]^R_{v,w} = 0$
      (3) $[\![\phi \to \psi]\!]^R_{v,w} = 1 \iff [\![\phi]\!]^R_{v,w} = 0$ or $[\![\psi]\!]^R_{v,w} = 1$
      (4) $[\![\Box\phi]\!]^R_{v,w} = 1 \iff$ for every w' $R(w, w') : [\![\phi]\!]^R_{v,w'} = 1$
      (5) $[\![\Diamond\phi]\!]^R_{v,w} = 1 \iff$ for some w' $R(w, w') : [\![\phi]\!]^R_{v,w'} = 1$
      ⋮
  - **Consequence**:
    - $\phi_1, \ldots, \phi_n \vDash \psi \iff$ For every $\mathcal{M}, w$: $[\![\psi]\!]^R_{v,w} = 1$ if $[\![\phi_1]\!]^R_{v,w} = \cdots = [\![\phi_n]\!]^R_{v,w} = 1$
      - $\mathcal{M} = \langle\langle R, W\rangle, v\rangle$ and $w \in W$

2.1 Motivating Kripke Semantics

- Why do we need R and W?
- Let's try to do without them!
  - Aren't valuations possible worlds?
  - The say how every atomic fact is settled
- Without W and R:
  - $[\![\Box\phi]\!]_v = 1 \iff$ for every $v' : [\![\phi]\!]^R_{v'} = 1$ (Carnap 1956: §41)
  - This only captures logical necessity:
    - Necessities which are true at every world if they are true at any world!
    - But what about nomological necessity?
      - What is necessary according to the physical laws of our world may not be true of other worlds
    - If ϕ is logically necessary, it is true!
    - What about deontic necessity?
      - There are many things which are morally required, but don't come true
      - Yet, this semantics says that $\Box\phi \to \phi$!
  - R allows us to restrict the space of possibilities considered for the truth of $\Box\phi$
    - It thereby allows the formulation of necessity claims weaker than logical necessity
- Without W (using valuations instead):
  - $[\![\Box\phi]\!]^R_v = 1 \iff$ for every v' $R(v, v') : [\![\phi]\!]^R_{v'} = 1$
  - There's a problem about having enough worlds
  - Suppose there's only one possible object a
    - It can either be an apple $v_1(\mathsf{A}) = 1$ or not $v_0(\mathsf{A}) = 0$
  - Suppose you know A in v_1: $[\![\Box\mathsf{A}]\!]^R_{v_1} = 1$
  - But you don't know that you know, so: $[\![\Diamond\neg\Box\mathsf{A}]\!]^R_{v_1} = 1$
    - Picture the accessibility relation!
    - v_1 can't see a $\neg\mathsf{A}$-world, but it must see a world w which can see a $\neg\mathsf{A}$-world
      - w must be an A world, but v_1 is the only A-world and it can't see a $\neg\mathsf{A}$-world
  - We need a third world to make $\Box\mathsf{A} \wedge \Diamond\neg\Box\mathsf{A}$ true!

2.2 Axioms and Systems

- Different constraints on R yield different logics for $\Box$ and $\Diamond$
- Here's a roadmap of interesting constraints and logics

Relation Types	Definition
Reflexive	$\forall w : R(w,w)$
Symmetric	$\forall w, w' : R(w',w)$ if $R(w,w')$
Serial	$\forall w, \exists w' : R(w,w')$
Transitive	$\forall w, w', w'' : R(w,w'')$ if $R(w,w')$ & $R(w',w'')$

Name	Axiom
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
T	$\Box\phi \to \phi$
B	$\phi \to \Box\Diamond\phi$
D	$\Box\phi \to \Diamond\phi$
4	$\Box\phi \to \Box\Box\phi$
E	$\Diamond\phi \to \Box\Diamond\phi$

System	Axioms	Accessibility
K	K	Any $W \times W$
T	KT	Reflexive
B	KTB	Reflexive & Symmetric
D	KD	Serial
S4	KT4	Reflexive & Transitive
S5	KTE	Symmetric & Transitive

2.3 Strict Conditionals

- Problems for material conditional:

  Material Negation $\neg(\mathsf{P} \to \mathsf{Q}) \vDash \mathsf{P}$

  Antecedent Negation $\neg\mathsf{P} \vDash \mathsf{P} \to \mathsf{Q}$

  Contraposition $\mathsf{P} \to \mathsf{Q} \vDash \neg\mathsf{Q} \to \neg\mathsf{P}$

- Facts about strict conditional:

  Material Negation Invalid $\neg\Box(\mathsf{P} \to \mathsf{Q}) \nvDash \mathsf{P}$

  Antecedent Negation Invalid $\neg\mathsf{P} \nvDash \Box(\mathsf{P} \to \mathsf{Q})$

  Contraposition $\Box(\mathsf{P} \to \mathsf{Q}) \vDash \Box(\neg\mathsf{Q} \to \neg\mathsf{P})$

  Necessary Antecedent Negation $\Box\neg\mathsf{P} \vDash \Box(\mathsf{P} \to \mathsf{Q})$

- Lewis and Stalnaker will raise more problems in Week 4.

2.4 Philosophical Issues

- *Actualism*: there's only one possible world, the actual one!
  - What about it from bit physics (Wheeler)? Too Wacky?

- *Reduction / Truth-making*: if modal truths were reducible to non-modal ones, you would expect possible worlds and valuations to be the same thing
  - How can our world make modal claims true, if modal logic's truth conditions obtain?
    - It requires our world to bear relations to other worlds
    - And that doesn't seem like something that just our world can do!
- *Instrumentalism*:
  - W doesn't really contain possible worlds, remember, we don't make any assumption about it's elements
    - Just that they are there!
- *Communication*:
  - On the modal logic analysis, modal claims only have truth-values relative to v, w, R
  - But where do these come from?
  - Context, you might say, but what exactly are contexts such that they would provide *these* things?
  - Can this capture the idea that different kinds of uses of a modal word exist in the same language?

3 Modal Logic and Natural Language

Simple Modal Logic Hypothesis The meaning of every modal expression in natural language can be expressed in terms of only two properties

(a) Whether it is a necessity or possibility modal, and

(b) Its accessibility relation, R.

3.1 Kinds of Modality

- Some basic kinds:

  Nomological (Natural Law) The feather and bowling ball must fall at the same speed.

  Metaphysical I could have been a doctor.

  Epistemic (Knowledge/Belief) It must be raining.

  Deontic (Moral Duty) John must be punished.

  Ability Will can eat a bottle of glue.

  Bouletic (Desire) I must have some coffee.

- Different accessibility relations for each?

- Treatment *must* and *should* in Portner:
  - Doesn't this make the difference between *must* and *should* the same as that between different flavors of modality?

3.2 Limitations of Modal Logic Analysis

- Within the kinds of modality, we find specific modifiers:
  - In view of the traffic laws, you must not speed.
- Mixtures of different accessibility relations in a conjunction:
  - I ought to donate to charity, and I might do so
- How do these parameters connect to what we ordinarily think of the context of an utterance?

References

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